2017

Set I

Section A *(Question numbers 1 to 10 carry 1 mark each)*

Question: 1

If a line makes angle 90°, 60° and 30° with the positive direction of x, y and z respectively, find its direction consines.

Answer:

Let the dc's of the lines be l, m, n. Then

$l = \cos 90^0$

$= 0$

$m = \cos 60^O$

$= \frac{1}{2}$

$n = \cos 30^O$

$= \frac{\sqrt{3}}{2}$

Question: 2

Check the continuity of the function f (x) = 2x + 3 at x = 1.

Answer:

Since f(x) is defined at x = 1

$\therefore\ f(1) = 2(1) + 3$
$= 5$

$\therefore \lim_{x\to 1} f(x) = \lim_{x\to 1}(2x+3)$

$= 2(1)+3 = 5$

$\therefore \lim_{x\to 1} f(x) = 5$

$= f(1)$

Hence, f is continuous at x = 1.

Question: 3

Using differentials, find the approximate value of $\sqrt{26}$.

Answer:

$\text{Let } f(x) = \sqrt{x} \Leftrightarrow f'(x) = \frac{1}{2\sqrt{x}}.$

$\text{For } = \sqrt{26} = \sqrt{25+1},$

take $x_0 = 25$
and h = 1.

$\therefore f(x_0) = f(25) = \sqrt{25} = 5;$

$f'(x_0) = \frac{1}{2x5}$

$= 0.1$

Using the formula for approximation,

$f(x_o + h) = f(x_0) + hf'(x_o)$, we have

$\Leftrightarrow \sqrt{26} = 5 + 1x0.1 = 5.1.$

Question: 4

Find the values of a, b, c and d from the following equations. $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$

Answer:

By equality of two matrices, we have

2a+b=4

a-2b=-3

5c-d=11

4c+3d=24

on solving the above equations, we get a=1,b=2, c=3 and d=4.

Question: 5

Evaluate: $\int \sin^4 x \, dx.$

Answer:

$\int \sin^4 x \, dx = \int \left(\sin^2 x\right)^2 dx$

$= \int \left[\frac{1-\cos 2x}{2}\right]^2 dx$

$= \frac{1}{4} \int (1-2\cos 2x+ \cos^2 2x) \, dx$

$= \frac{1}{4} \int \left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right] dx$

$= \frac{1}{8} \int \left[3 - 4\cos 2x + \cos 4x\right] dx$

$= \frac{1}{8}\left(3 - 2\sin 2x + \frac{\sin 4x}{4}\right) + c.$

Question: 6

If $a^\wedge$ and $b^\wedge$ are two unit vectors and θ is angle between them, show that $\sin\frac{\theta}{2} = \frac{1}{2}\left|a^\wedge - b^\wedge\right|$.

Answer:

$\left|a^\wedge - b^\wedge\right|^2 = \left(a^\wedge - b^\wedge\right).\left(a^\wedge - b^\wedge\right)$

$\left|\hat{a}-\hat{b}\right|^2 = \left|\hat{a}\right|^2 + \left|\hat{b}\right|^2 - 2\left|\hat{a}\right|\left|\hat{b}\right|\cos\theta$

$= 1^2 + 1^2 - 2(1)(1)\cos\theta = 2\text{-}2\cos\theta$

$2(1-\cos\theta)$

$2.2\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2}$

$\therefore \sin^2\frac{\theta}{2} = \frac{1}{4}\left|\hat{a}-\hat{b}\right|^2$

$\sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a}-\hat{b}\right|$

Question: 7

Find the values of a and b for which the following holder: $\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Answer:

Here given, $\begin{bmatrix} a-b & \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 2a-b \\ -2a-2b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

By definition of equality of matrices

2a-b = 5 (i)

-2a –2b = 4 (ii)

On adding (i) and (ii), we get -3b = 9 $\Rightarrow$ b= -3

Putting b= -3 in eq. (i), we get 2a+3 = 5 $\Rightarrow$ a =1

Question: 8

The dot product of a vector with the vectors $\hat{i}+\hat{j}=3\hat{k}, \hat{i}+3\hat{j}=2\hat{k}$ and $2\hat{i}+\hat{j}+4\hat{k}$ are 0, 5 and 8 respectively. Find the vector.

Answer:

Let the required vector be $(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+\hat{j}+3\hat{k})=0 \Rightarrow x+y-3z=0$

$\Rightarrow$ x + y - 3z = 0 = $(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+\hat{j}+3\hat{k}) = 5$

$\Rightarrow$ x + 3y - 2z = 5 = $(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+\hat{j}+3\hat{k}) = 8$

$\Rightarrow$ 2x + y + 4z = 8

Solving these equations, we get, X = 1, y = 2, z = 1

$\therefore$ The required vector is $\hat{i}+2\hat{j}+\hat{k}$

Question: 9

Write the following functions in the simplest from: $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}, x > \pi.$

Answer:

$$\text{Let } y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$y = \tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$$

$$y = \tan^{-1}\tan\frac{x}{2}$$

$$y = \frac{x}{2}.$$

Question: 10

Evaluate: $\{\dfrac{x\,dx}{(x+2)(3-2x)}$

Answer:

$$\text{Let } I = \int\frac{xdx}{(x+2)\quad(3-2x)}$$

Resolve into partial fractions, $\displaystyle\int\left[-\frac{2}{7\quad(x+2)}+\frac{3}{7\quad(3-2x)}\right]dx$

Integrate w.r.t x $\displaystyle I=\frac{2}{7}\log|x+2|+\frac{3\log|3-2x|}{-2}+c$

$$=\frac{2}{7}\log|x+2|+\frac{3}{14}\log|3-2x|+c$$

Section B *(Questions numbers 11 to 22 carry 4 marks each)*

Question: 11

Veryfi Rolle's Theorem for the function $f(x) = x^2 - x - 12$ in $[-3,4]$.

Answer:

Here, f(x) = x – x -12 in [-3, 4]

Being a polynomial, f (x) is cont. in [-3, 4] and differentiable in [-3, 4]

Also $f(-3) = (-3)^2 - (-3) - 12 = 9 + 3 - 12 = 0$ and $f(4) = 4^2 - 4 - 12 = 16 - 4 - 12 = 0$

$\Rightarrow f(-3) = f(4)$.(=0)

$\therefore$ All the conditions of Roolle's theorem are satisfied.

$\Rightarrow \$\ c\ \hat{I}\ ]-3,4[\ \text{s.t. } f'(c) = 0$

Now f'(c) = 2c -1.

$$f'(c) = 0 \Rightarrow 2c - 1 = 0 \Rightarrow c = \frac{1}{2},$$

Clearly $c = \frac{1}{2} \in [-3, 4]$

$\therefore$ Rolle's theorem is verified.

Question: 12

Find the intervals in which the function $f(x =)2x^3 - 9x^2 + 12 + 30$ is (i) increasing. (ii) dereasing.

Answer:

Here, $f(x) = 2x^3 - 9x^2 + 12x + 30$

$\Rightarrow f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x-1)(x-2)$

For f to be increasing, we have

f'(x) ñ 0 ⇒ (x - 1)(x - 2)ño

⇒ x - 1ñ0 and x - 2ñ0

or x - 1á0 and x - 2á0

⇒xñ2 or xá1.

$\therefore$ f will be increasing in $]-\infty, 1[\cup]2, \infty[$.

Similarly f will be decreasing in $]1, 2[$

Question: 13

Evaluate : $\int \log(1 + x^2)dx$.

Answer:

Let $1 = \int \log(1 + x^2)dx$

$= \int 1.\log(1 + x^2)dx$

Taking I as second function and applying the rule of integration by parts, we get

$I = x\log(I + x^2) - \int x.\frac{1}{1 + x^2}.2x\,dx$

$= x\log(1 + x^2) - 2\int \frac{x^2}{1 + x^2}dx$

$= x\log(1 + x^2) - 2\int \left[1 - \frac{1}{1 + x^2}\right]dx$

$x\log(1+x^2) - 2(x - \tan\text{-}1x) + c.$

OR

$y = x^{\sin x} + (\sin x)^x$, find $\frac{dy}{dx}$.

Answer:

Given : $y = x^{\sin x} + (\sin x)^x = u + v$ (1)

where $u = x^{\sin x}$ and $v = (\sin x)^x$

$\Rightarrow$ log u = sin x logx

$$\Rightarrow \frac{1}{u}.\frac{dx}{dx} = \cos x \log x + \frac{\sin x}{x}$$

$$\Rightarrow \frac{dx}{dx} = x^{\sin x}\left[\cos x \log + \frac{\sin x}{x}\right] \qquad (2)$$

Also v = (sin x)x

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = 1.\log \sin x + x.\frac{1}{\sin x}.\cos x$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^x\left[\log \sin x + x \cot x\right]$$

$$\text{From}(1), \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\sin x\left[\cos x \log x + \frac{\sin x}{x}\right]}$$

+ (sin)x [logsin x + xcotx] [using (2) and (3)]

Question: 14

Show that the function: $f(x) = \begin{cases} \frac{\sin x}{2^x} + \cos x, x ^1 0, x = 0 \end{cases}$ is continuous at x = 0

Answer:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin x}{x} + \cos x\right) = 1+1=2$$

Also $f(0) = 2$

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

Hence $f(x)$ is continuous at $x = 0$

OR

Find the value of K so that the function $f(x) = \begin{cases} Kx^2, \text{ if } x \le 2 \; 3, \text{ if } x > 2 \end{cases}$ is continuous at x=2

Answer:

$\therefore f(x)$ is continuous at $x = 2$

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = f(2)$$

$$\therefore \lim_{x\to 2-} Kx^2 = \lim_{x\to 2+} 3$$

$$K(2)^2 = 3$$

$$4K = 3 \qquad \Rightarrow K = \frac{3}{4}$$

Question: 15

Prove that $3\sin^{-1}x = \sin^{-1}\left(3x-4x^{3}\right), 4 \text{ Î} \left[-\frac{1}{2},\frac{1}{2}\right]$

Answer:

R.H.S= $\sin^{-1}\left(3x-4x^{3}\right)$

put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

R.H.S. $= \sin^{-1}\left(3\sin\theta - 4\sin^{3}\theta\right)$

$\sin^{-1}\left(\sin 3\theta\right)$

$= 3\theta$

$= 3\sin^{-1}x$

$\therefore$ R.H.S = L.H.S.

Question: 16

If the sum of the mean and variance of a binomial Distribution for 5 trials be 1.8, find the distribution.

By hypothesis, Np npq = 1.8 and = 5

$\Rightarrow 5\left(\pi + \pi\theta\right) = 1.8 \quad \Rightarrow \pi + \pi\left(1-\pi\right) = 0.36$

$\Rightarrow \pi^{2} - 2\pi + 0.36 = 0$

$\Rightarrow \pi = \dfrac{2+\sqrt{4-1.44}}{2}$

$= \dfrac{2+1.6}{2}$

$\Rightarrow \pi = \dfrac{0.4}{2} = \dfrac{1}{5}$

$\left(\text{P}\varepsilon j\varepsilon\chi\tau : \pi = 1.8\right)$

Binomial Distribution $\theta + \pi)^{n} = \left(\frac{4}{5}+\frac{1}{5}\right)^{5}$.

Question: 17

Find the angle between the pair of lines given by $r^{«} = 3i^{Ù} + 2j^{Ù} - 4k^{Ù}$ $+k\left(i^{Ù} + 2j^{Ù} + 2k^{Ù}\right)$ and $r = 5i^{Ù} - 2j^{Ù} + u\left(3i^{Ù} + 2j^{Ù} + 6k^{Ù}\right)$

Answer:

On compairing by $r^{\leftrightarrow} = a^{\overleftrightarrow{}} + kb \rightarrow$

Here, $b_{1} = i^{\wedge} + 2j^{\wedge} + 2k^{\wedge}\ b_{2}$

$= 3\hat{i} + 2\hat{j} + 6\hat{k}$

The angle θ between the two lines is given by

$$\cos\theta \left|\frac{\vec{b_1}.\vec{b_2}}{|b_1||\vec{b_2}|}\right| = \left|\frac{(\hat{i} + 2\hat{j} + 2\hat{k}).(3\hat{i} + 2\hat{j} + 6\hat{k})}{\sqrt{1+4+4}\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3X7}\right|$$

$$= \frac{19}{21}$$

$$\therefore \theta\cos^{-1}\left(\frac{19}{21}\right)$$

OR

Differentiate the following function w.r.t.x : $x^{\sin x} + (\sin x)^{\cos x}$.

Answer:

Let $y = x^{\sin x} + (\sin x)^{\cos x}$ .put $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$.

$\therefore y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} = \frac{dv}{dx}$ consider $u = x^{\sin x}$

Taking logarithms on both sides we get log u= sin x log x

Differentiate with respect to x,we get $\frac{1}{u}\frac{du}{dx} = \sin x.\frac{1}{x} + \log x.\cos x$

$$\frac{du}{dx} = u\left(\frac{\sin x}{x} + \cos x \log x\right)$$

$$\frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

Now consider $v = (\sin x)^{\cos x}$

Taking logarithms on both sides,we get $\log v = \cos x.\log\sin x$

Differential with respect to x, we get $\frac{1}{v}\frac{dv}{dx} = \cos x\frac{1}{\sin x}\cos x + \log\sin x(-\sin x)$

$$\frac{dv}{dx} = v[\cos x\cot x - \sin x\log\sin x]$$

$$\therefore \frac{dv}{dx} = (\sin x)^{\cos x}[\cos x\cot x - \sin x\log\sin x]$$

Hence $\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x\log x\right] + (\sin x)^{\cos x}[\cos x\cot x - \sin x\log\sin x]$

Question: 18

A diet to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs. 4 per unit and F_2 costs Rs 6 per unit. One unit of food F1 contain 3 unit of vitamin A and 4 units of minerals.

One units of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem and find graphically the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

Answer:

Let the diet contain x units of food F_1 and y units of food F_2.
Now construct the following table:

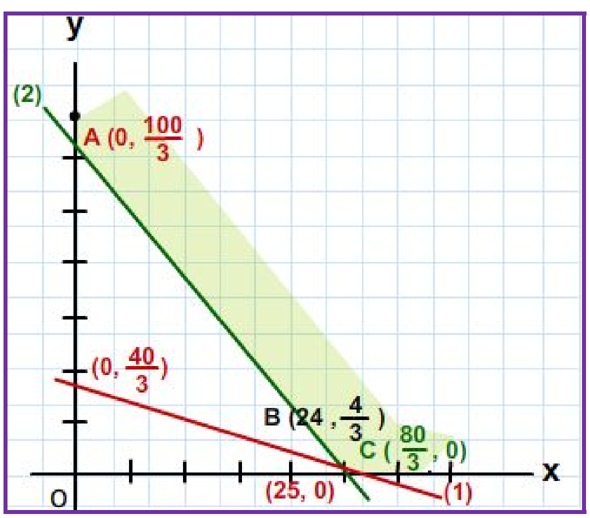
Food Type	Vitamin A	Minerals	Cost
F2(x) F2 (y)	3 6	4 3	Rs. 4 Rs. 6
Total	≥ 80	≥ 100	

Required L.P.P is
Minimum cost Z – 4x + 6y
Subject to constraints
$3x + 6y \geq 80$
$4x + 3y \geq 100$
$x, y > 0$
Table for $3x + 6y = 80$

X	0	$\frac{80}{3}$
Y	$\frac{40}{3}$	0

Table for $4x + 3y = 100$

x	0	25
y	100 3	0



$\therefore A\left(0,\frac{100}{3}\right), B\left(24,\frac{4}{3}\right)$ and $C\left(\frac{80}{3},0\right)$

Now evaluate Z at corner points

Corner points	Z = 4x + 6y
$A\left(0,\frac{100}{3}\right)$	$Z = 0 + 6 \times \frac{100}{3} = 200$
$B\left(24,\frac{4}{3}\right)$	$Z = 4 \times 24 + 6 \times \frac{4}{3} = 104$ (min)
$C\left(\frac{80}{3},0\right)$	$Z = 4 \times \frac{80}{3} + 0 = \frac{320}{3} = 106.6$

$\therefore$ The minimum cost is Rs. 104 when x = 24 units and $y = \frac{4}{3}$ units

Question: 19

A family has 2 children. Find the probability that both are boys, if it is known that

i. At least one of the children is a boy
ii. The elder child is a boy.

Answer:

Let event A is that the family has two boys

i. event B: At least one is a boy

P(both boys, given that at least one is a boy) P (A/B)

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P\{(B,B)\}}{P\{(B,G),(G,B),(B,B)\}}$$

$$\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(ii) event C: the elder child is a boy

P(both boys, given that at elder child is a boy) = $P(\frac{A}{C})$

$$\frac{P(A \cap C)}{P(C)} = \frac{P\{(B,B)\}}{P\{(B,G),(B,B)\}}$$

$$\frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

Question: 20

If $\vec{a} =$ I + j + k, b = 4i – 2j + 3k and c = I – 2j + k, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$

Answer:

$2\vec{a} - \vec{b} + 3\vec{c} = \left(2\hat{i}+2\hat{j}+2\hat{k}\right) - \left(4\hat{i}-2\hat{j}+3\hat{k}\right) + \left(3\hat{i}-6\hat{j}+3\hat{k}\right) = \left(\hat{i}-2\hat{j}+2\hat{k}\right)$

$\left|2\vec{a} - \vec{b} + 3\vec{c}\right| = 3$

$\therefore$ Required vector = $2\hat{i}-4\hat{j}+4\hat{k}$

OR

Let $\vec{a} = -\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} - 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d} = 18$

Answer:

A vector perpendicular to $\vec{a}$ and $\vec{b} = \vec{a} \times \vec{b} = 32\hat{i}-\hat{j}-14\hat{k}$

Let $\vec{d} = \lambda\left(32\hat{i}-\hat{j}-14\hat{k}\right)$

$\therefore \vec{c}.\vec{d} = 18 \Rightarrow \lambda\left(64+1-56\right) = 18 \Rightarrow \lambda = 2$

$\therefore \vec{d} = 64\hat{i}-2\hat{j}-28\hat{k}$

Question: 21

Find the equation of the plane passing through the point (-1, 3, 2) and perpendicular to each of the planes x + 2y + 3z = 5 and 3x + 3y + z = 0 .

Answer:

The equation of plane through (-1, 3, 2) can be expressed as

A(x + 1) + B(y - 3) + C(z - 2) = 0 … (i)

As the required plane is perpendicular to x + 2y + 3z = 5 and 3x + 3y + z = 0, we get

A + 2B + 3C = 0

3A + 3B + C = 0

$\Rightarrow \frac{A}{2-9} = \frac{B}{9-1} = \frac{C}{3-6} \Rightarrow \frac{A}{-7} = \frac{B}{8} = \frac{C}{-3}$

$\therefore$ Direction ratios of normal to the required plane are -7, 8, –3.

Hence equation of the plane will be

-7(x + 1) + 8(y - 3) - 3(z - 2) = 0

$\Rightarrow$ -7x - 7 + 8y - 24 - 3z + 6 = 0

or 7x - 8y + 3z + 25 = 0

Question: 22

Write the following functions in the simplest form $\tan^{-1}\left(\sqrt{1+x^2}+x\right)$

Answer:

Let $y = \tan^{-1}\left(\sqrt{1+x^2}+x\right)$

Put $x = \tan\ \theta \Rightarrow \theta \tan^{-1} x$

$y=\tan^{-1}\left(\sqrt{\tan^2\theta}+\tan\theta\right)$

$y = \tan^{-1}\left(\sqrt{\sec^2\theta}+\tan\theta\right)$

$y=\tan^{-1}\left(\sqrt{\sec\theta}+\tan\theta\right)$

$y=\tan^{-1}\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)$

$y= \tan^{-1}\left(\frac{1+\sin\theta}{\cos\theta}\right)$

Section C *(Question numbers 23 to 29 carry 6 marks each)*

Question: 23

Find the angle between the pair of lines given by $r^{«} = 3i^{Ù} + 2j^{Ù} - 4k^{Ù}$ $+k\left(i^{Ù}+2j^{Ù}+2k^{Ù}\right)$ and $r = 5i^{Ù} - 2j^{Ù} + u\left(3i^{Ù}+2j^{Ù}+6k^{Ù}\right)$

Answer:

On compairing by $r^{\leftrightarrow} = a^{\overleftrightarrow{}} + kb \rightarrow$

Here, $b_1 = i^{\wedge} + 2j^{\wedge} + 2k^{\wedge}$ b_2

$= 3i^{\wedge} + 2j^{\wedge} + 6k^{\wedge}$

The angle θ between the two lines is given by

$\cos\theta\left|\frac{b_1^{\rightarrow}.b_2^{\rightarrow}}{|b_1||b_2^{\rightarrow}|}\right| = \left|\frac{\left(i^{\wedge}+2j^{\wedge}+2k^{\wedge}\right).\left(3i^{\wedge}+2j^{\wedge}+6k^{\wedge}\right)}{\sqrt{1+4+4\sqrt{9+4+36}}}\right|$

$=\left|\frac{3+4+12}{3X7}\right|$

$=\frac{19}{21}$

$\therefore \theta\cos^{-1}\left(\frac{19}{21}\right)$

Question: 24

Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Find the point of intersection also.

Answer:

Two given lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar if

$$\Delta=\begin{vmatrix} x_2-x_1, & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

Here the lines are $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$

and $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1}$

$$\therefore \Delta=\begin{vmatrix} 4-1 & 1-2 & 0-3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}=\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{vmatrix}=0$$

$\left(\text{By} R_3 \to R_3-R_1-R_2\right)$

$\Rightarrow$ Lines (1) and (2) are coplanar.

Now any point on (1) is $(1+2r, 2+3r, 3+4r)$

This will lie on (2) if

$$\frac{1+2r-4}{5}=\frac{2+3r-1}{2}=\frac{3+4r}{1}.$$

$$\Rightarrow \frac{2r-3}{5}=\frac{3r+1}{2}=\frac{4r+3}{1}$$

Solving for r, we get r = -1

Question: 25

Define the line of shortest distance between two skew lines. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by

$\vec{r}=\left(2\hat{j}-3\hat{k}\right)+\lambda\left(2\hat{i}-\hat{j}\right)$ and $\vec{r}=\left(4\hat{i}+3\hat{k}\right)+\mu\left(3\hat{i}+\hat{j}+\hat{k}\right)$.

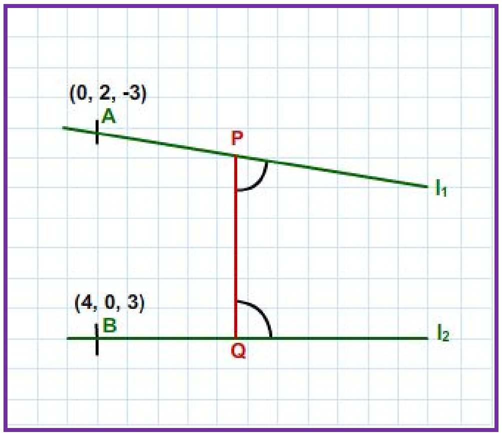
$$\left.\begin{array}{l} \vec{r}=\left(2\hat{j}-3\hat{k}\right)+\lambda\left(2\hat{i}-\hat{j}\right), \\ \vec{r}=\left(4\hat{i}+3\hat{k}\right)+\mu\left(3\hat{i}+\hat{j}+\hat{k}\right) \end{array}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

We can write this as

$$\left.\begin{array}{l}\vec{r} = 2\lambda\hat{i} + (2-\lambda)\hat{j} - 3\hat{k} \\ \vec{r} = (4+3\mu)\hat{i} + \mu\hat{j} + (3+\mu)\hat{k}\end{array}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

Answer:

Let the two lines be represented by l_1 and l_2. Now l_1 passes through (0, 2 -3) and is parallel to $2\hat{i} - \hat{j}$. Also l2 passes through, (4, 0, 3) and is parallel to $3\hat{i} + \hat{j} + \hat{k}$.



From (ii), we get two variable points, one on each line, say $P(2\lambda, 2-\lambda, -3)$ On l_1 and $Q(4+3\mu, \mu, 3+\mu)$ on l_2.

$$\vec{PQ} = \left(\mu(4+3\mu-2\lambda)\hat{i} + -2+\lambda\right)\hat{j} + (3+\mu+3)\hat{k}.$$

If $\vec{PQ}$ is the shortest distance vector, then it should be ^ to both $2\hat{i} - \hat{j}$ and $3\hat{i} + \hat{j} + \hat{k}$.

$\Rightarrow (4+3\mu-2\lambda).2 + (\mu-2+\lambda).$

$(-1) + (6+3\mu).0 = 0$ and $(4+3\mu-2\lambda).3 + (\mu-2+\lambda).$

$1 + (6+3\mu).1 = 0$

$\Rightarrow \lambda - \mu = 2$ and $5\lambda - 11\mu = 16$

$\Rightarrow \lambda = 1,\ \mu = 1$

$\therefore$ Points P and Q are P(2,1-3) and (1,-1,2)

$\therefore$ Shortest distance $= PQ = \sqrt{(2-1)^2 + (1+1)^2 + (-3+2)} = \sqrt{30}$

Also the vector equation of the shortest distance PQ is

$\vec{r} = (\text{Position vector of } P) + t\overrightarrow{PQ}$

$= (2\hat{i} + \hat{j} + 3\hat{k}) + t(-\hat{i} - 2\hat{j} + 5\hat{k})$

Question: 26

Observe the inverse of the following matrix using elementary operations. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Answer:

Since A = IA

$$\text{or}\begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

$$\text{Applying}R_1 \leftrightarrow R_2 \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}A$$

$\text{Applying}R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & -5 & -8\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix}A$$

$$\text{Applying}R_1 \rightarrow R_1 - 2R_2 \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix}A$$

$$\text{Applying}R_1 \rightarrow \frac{1}{2}R_3 \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\1 & 0 & 0\\\frac{5}{2} & -\frac{3}{4} & \frac{1}{2}\end{bmatrix}$$

Question: 27

Find the equation of the plane passing through the point (I, 1,1) and perpendicular to each of the following palnes: x +2y + 3z = 7 and 2x – 3y + 4z = 0

Answer:

Equation of plane through (1,1,1) is lx-1) + m(y-1) + n(z-1) = 0 (1)

Now we have to find l,m,n when (1) is perpendicular to both the planes

X+2y+3z = 7

2x-3y+4z = 0

$\therefore$ l x 1 + m x 2 + n x 3 = 0

l x 2 + m x (-3) + n x 4 = 0 $\Rightarrow$ l + 2m + 3n = 0

$2l - 3m + 4n = 0 \Rightarrow \dfrac{l}{8+9}=\dfrac{m}{6-4}=\dfrac{n}{-3-4} \Rightarrow \dfrac{1}{17}=\dfrac{m}{2}=\dfrac{n}{-7}$

$\therefore$ Equation of the required plane is 17 (x-1) + 2(y-1) – 7(z-1) = 0 $\Rightarrow$ 17x + 2y –7z = 12

Question: 28

Using properties of determinants, prove the following

$$\begin{vmatrix} a & B^2 & y \\ a^2 & B^2 & y^2 \\ B+y & y+a & a+B \end{vmatrix} = (a-B)(B-y)(y-a)(a+B+y).$$

Answer

Let $\varsigma = \begin{vmatrix} a & B & y \\ a^2 & B^2 & y^2 \\ B+y & y+a & a+B \end{vmatrix}$

Applying $R_3 \to R_3 + R_1$

$$\Delta = \begin{vmatrix} a & B & y \\ a^2 & B^2 & y^2 \\ a+B+y & B+y+a & y+a+B \end{vmatrix}$$

$$= (a+B+y)\begin{vmatrix} a & B & y \\ a^2 & B^2 & y^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$c_2 \to c_2 - c_1, c_3 \to c_3 - c_1$

$$= (a+B+y)\begin{vmatrix} a & B-y & y-a \\ a^2 & B^2-a^2 & y^2-a^2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (a+B+y)\begin{vmatrix} a & B-a & y-a \\ a^2 & (B+a)(B-a) & (y-a)(y+a) \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (a+B+y)(B-a)(y-a)\begin{vmatrix} a & 1 & 1 \\ a^2 & B+a & y+a \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (a+B+y)(B-a)(y-a)\left[1(y+a-B-a)\right]$$

$$= (a+B+y)(B-y)(y-a)(y-a)$$

$$= (a-B)(B-y)(y-a)(a+B+y)$$

Question: 29

Find the equation of the plane passing through the point $(-1,-1,2)$ and perpendicular to each of the following planes : $2x+3y-3z=2$ and $5x-4y+z=6$.

Answer:

Equation of plane passing through $(-1,-1,2)$ is $a(x+1)+b(y+1)+c(z-2)=0$

Since plane (1) is perpendicular to the following planes

$2x+3y-3z=2$

$5x-4y+z=6$

$2a+3b-3c=0$

$\left[\therefore a_1a_2+b_1b_2+c_1c_2=0\right]$

$5a-4b+c=0$

Eliminating a, b, c from (1), (4) and (5), we get

$$\begin{vmatrix} x+1 & y+1 & z-2 \\ 2 & 3 & -3 \\ 5 & -4 & 1 \end{vmatrix}=0$$

$(x+1)(3-12)-(y+1)(2+15)+(z-2)(-8-15)=0$

$-11(x+1)-17(y+1)-23(z-2)=0$

$-11x-11-17y-17-23z+46=0$

$-11x-17y-23z+18=0$

$11x+17y+23z-18=0$