
2016

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Section A (*Question numbers 1 to 6 carry 1 mark each*)



Question: 1

If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, find α satisfying $0 < \alpha < \frac{\pi}{2}$ when $A + A^T = \sqrt{2}I_2$: where A^T is transpose of A .

Answer:

$$A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \quad 0 < \alpha < \frac{\pi}{2}$$

$$A + A^T = \sqrt{2}I_2$$

$$\begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} + \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2\cos \alpha & 0 \\ 0 & 2\cos \alpha \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$2 \cos \alpha = \sqrt{2}$$

$$\cos \alpha = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\alpha = \frac{\pi}{4}$$

Question: 2

If A is a 3×3 matrix and $|3A| = k|A|$, then write the value of k .

Answer:

$$|3A| = k|A|$$

$$|3A| = 27|A|$$

$$k = 27$$

Question: 3

For what values of k , the system of linear equations

$$x + y + z = 2$$

$$2x + y + z = 3$$

$$3x + 2y + kz = 4$$

has a unique solution ?

Answer:

For unique solution $|A| \neq 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$$C_2 \rightarrow C_2 - C_1 ; C_3 \rightarrow C_3 - C_1$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -3 \\ 3 & -1 & k-3 \end{vmatrix} \neq 0$$

Expansion along R_1

$$-(k-3) - 3 \neq 0$$

$$-k + 3 - 3 \neq 0$$



$$k \neq 0$$

Question: 4

Write the sum of intercepts cut off by the plane $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) - 5 = 0$, on the three axes.

Answer:

$$\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) - 5 = 0$$

In Cartesian form

$$2x + y - z - 5 = 0$$

$$2x + y - z = 5$$

$$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1$$

$$\frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1$$

Intercept cut off on the axes $\left(\frac{5}{2}, 5, -5\right)$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$a = \frac{5}{2} \quad b = 5 \quad c = -5$$

$$a + b + c = \frac{5}{2}$$

Question: 5

Find λ and μ if $(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = \vec{0}$

Answer:

$$(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = \vec{0}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 9 \\ 3 & -\lambda & \mu \end{vmatrix} = \vec{0}$$

$$\hat{i}(3\mu + 9\lambda) - \hat{j}(\mu - 27) + \hat{k}(-\lambda - 9) = \vec{0}$$

$$3\mu + 9\lambda = 0 \dots\dots\dots (i)$$

$$27 - \mu = 0 \dots\dots\dots (ii)$$

$$-\lambda - 9 = 0 \dots\dots\dots (iii)$$

By equation (ii) & (iii) $\mu = 27 \quad \lambda = -9$

λ, μ value satisfy the equation (i)

So $\mu = 27 \quad \lambda = -9$

Question: 6

If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector parallel to the vector $\vec{a} + \vec{b}$.

Answer:

$$\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$$



$$\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{a} + \vec{b} = (4\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 2\hat{j} + \hat{k})$$

$$= 6\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\text{unit vector parallel to } (\vec{a} + \vec{b}) = \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|}$$

$$= \frac{6\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{36 + 9 + 4}}$$

$$= \frac{6\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{49}}$$

$$= \frac{6}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}$$

Section: B (Question numbers 7 to 19 carry 4 mark each)

Question: 7

Solve for x: $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$

Answer:

$$\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$$

$$\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$$

$$\tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right) = \tan^{-1}\left(\frac{3x-x}{1-3x^2}\right)$$

$$\frac{2x}{1-(x^2-1)} = \left(\frac{2x}{1+3x^2}\right)$$

$$x(1+3x^2) = x(2-x^2)$$

$$x(1+3x^2-2+x^2) = 0$$

$$x(4x^2-1) = 0$$

$$x = 0; \quad 4x^2 - 1 = 0$$

$$x^2 = \frac{1}{4}$$

$$x = \pm \frac{1}{2}$$

$$x = 0, \pm \frac{1}{2}$$

OR

Prove that $\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right) - \tan^{-1}\left(\frac{4x}{1-4x^2}\right) = \tan^{-1}2x; |2x| < \frac{1}{\sqrt{3}}$

Answer:

$$\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right) - \tan^{-1}\left(\frac{4x}{1-4x^2}\right)$$



$$\tan^{-1} \left(\frac{\frac{6x-8x^3}{1-12x^2} - \frac{4x}{1-4x^2}}{1 + \frac{6x-8x^3}{1-12x^2} \times \frac{4x}{1-4x^2}} \right)$$

$$\tan^{-1} \left(\frac{(6x-8x^3)(1-4x^2) - 4x(1-12x^2)}{(1-12x^2)(1-4x^2) + (6x-8x^3)4x} \right)$$

$$\tan^{-1} \left(\frac{6x-24x^3-8x^3+32x^5-4x+48x^3}{1-4x-12x^2+48x^4+24x^2-32x^4} \right)$$

$$\tan^{-1} \left(\frac{32x^5+16x^3+2x}{16x^4+8x^2+1} \right)$$

$$\tan^{-1} \left(2x \frac{(16x^4+8x^2+1)}{16x^4+8x^2+1} \right)$$

$$\tan^{-1} 2x$$

L.H.S. = R.H.S.

Question: 8

A typist charges Rs.145 for typing 10 English and 3 Hindi pages, while charges for typing 3 English and 10 Hindi pages are Rs.180. Using matrices, find the charges of typing one English and one Hindi page separately. However typist charged only Rs.2 per page from a poor student Shyam for 5 Hindi pages. How much less was charged from this poor boy? Which values are reflected in this problem?

Answer:

$$\begin{bmatrix} 10 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [145]$$

$$\begin{bmatrix} 3 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [180]$$

$$10x + 3y = 145$$

$$3x + 10y = 180$$

By solving the equations we get

$$x = 10, y = 15$$

But Typist charge 2 Rs. per Page from a Poor student shyam

Amount taken by shyam = $2 \times 5 = 10$ Rs.

But from another person, he take for

5 Pages = 15×5

$$= 75 \text{ Rs.}$$

Amount differ by = $75 - 10$

$$= 65 \text{ Rs. Less.}$$

Sympathy are reflect this problem

Question: 9

$$\text{If } f(x) = \begin{cases} \frac{\sin(a+1)x + 2\sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \frac{\sqrt{1+bx} - 1}{x}, & x > 0 \end{cases} \text{ is continuous at } x=0, \text{ then find the values of } a \text{ and } b.$$

Answer:

At $x = 0$ function is continuous,



$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0)$$

R.H.L.

$$= \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} \frac{\sqrt{1+bh}-1}{h} \times \frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}$$

$$= \lim_{x \rightarrow 0} \frac{1+bh-1}{h(\sqrt{1+bh}+1)}$$

$$= \lim_{x \rightarrow 0} \frac{bh}{h(\sqrt{1+bh}+1)}$$

$$= \lim_{x \rightarrow 0} \frac{bh}{\sqrt{1+bh}+1}$$

$$= \frac{b}{2}$$

$$f(0) = 2$$

L.H.L.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h)$$

$$= \lim_{h \rightarrow 0} \frac{\sin(a+1)(0-h) + \sin(0-h)}{0-h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(-(a+1)h) + \sin(-h)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(a+1)h}{h} + \frac{2 \sin h}{h}$$

$$= (a+1) + 2$$

$$\begin{aligned} a+3 &= 2 & \frac{b}{2} &= 2 \\ a &= -1 & b &= 4 \end{aligned}$$

Question: 10

If $x \cos(a+y) = \cos y$ then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a+y) \frac{dy}{dx} = 0$

Answer:

$$x \cos(a+y) = \cos y$$

$$x = \frac{\cos y}{\cos(a+y)}$$

$$\frac{dy}{dx} = \frac{\cos(a+y)x(-\sin y) - \cos y(-\sin(a+y))}{\cos^2(a+y)}$$

$$\frac{dy}{dx} = \frac{-\sin y \cos(a+y) + \cos y \sin(a+y)}{\cos^2(a+y)}$$

$$\therefore \sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\frac{dy}{dx} = \frac{\sin a}{\cos^2(a+y)} \text{ So, } \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$



$$\sin a \frac{d^2 y}{dx^2} = 2 \cos(a+y)x(-\sin(a+y)) \frac{dy}{dx}$$

$$\sin a \frac{d^2 y}{dx^2} + 2 \cos(a+y)x(\sin(a+y)) \frac{dy}{dx} = 0$$

$$\sin a \frac{d^2 y}{dx^2} + \sin 2(a+y) \frac{dy}{dx} = 0$$

OR

Find $\frac{dy}{dx}$ if $y = \sin^{-1} \left[\frac{6x - 4\sqrt{1-4x^2}}{5} \right]$

Answer:

$$y = \sin^{-1} \left[\frac{6x - 4\sqrt{1-4x^2}}{5} \right]$$

$$y = \sin^{-1} \left[\frac{6x}{5} - \frac{4}{5} \sqrt{1-4x^2} \right]$$

$$y = \sin^{-1} \left[2x - \frac{4}{5} \sqrt{1-(2x)^2} \right]$$

$$\sin^{-1} p - \sin^{-1} q = \sin^{-1} (p\sqrt{1-q^2} - q\sqrt{1-p^2})$$

$$p = 2x \quad q = \frac{4}{5}$$

$$y = \sin^{-1} 2x - \sin^{-1} \frac{4}{5}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \times 2 \cdot 1 - 0$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$$

Question: 11

Find the equation of tangents to the curve $y = x^3 + 2x - 4$, which are perpendicular to line $x + 14y + 3 = 0$.

Answer:

$$y = x^3 + 2x - 4$$

$$\left(\frac{dy}{dx} \right)_{c_1} = 3x^2 + 2 \dots\dots\dots (i)$$

$$\text{Equation of tangent: } y - y_1 = m(x - x_1) \dots\dots\dots (ii)$$

is \perp to $x + 14y + 3 = 0$

$$m = \left(\frac{dy}{dx} \right)_{c_1} = 3x^2 + 2$$

$$14y = -x - 3$$

$$y = \frac{-x-3}{14}$$

$$m \times \frac{-1}{14} = -1$$



$$m = 14$$

$$3x^2 + 12 = 14$$

$$3x^2 = 12$$

$$x^2 = 14 \quad x = \pm 2$$

$$\text{if } x = 2 \quad x = -2$$

$$y = 2^3 + 2 \cdot 2 - 4 \quad y = -8 - 4 - 4$$

$$y = 8 + 4 - 4 \quad y = -16$$

$$y = 8$$

$$P_1 (2, 8) \quad P_2 (-2, -16)$$

equation of tangent at $P_1 (2, 8)$

$$y - 8 = 14 (x - 2)$$

$$y - 8 = 14x - 28$$

$$14x - y = 20$$

equation of tangent at $P_2 (-2, -16)$

$$y + 16 = 14 (x + 2)$$

$$14x - y = 16 - 28$$

$$14x - y = -12$$

Question: 12

Find: $\int \frac{(2x-5)e^{2x}}{(2x-3)^3} dx$.

Answer:

$$I = \int \frac{(2x-5)e^{2x}}{(2x-3)^3} dx$$

$$= \int e^{2x-3} \cdot e^3 \left[\frac{(2x-3)-2}{(2x-3)^3} \right] dx$$

$$= \int e^{2x-3} \cdot e^3 \left[\frac{1}{(2x-3)^2} - \frac{2}{(2x-3)^3} \right] dx$$

Let $f(x) = \frac{1}{(2x-3)^2}$ whose, $2x-3 = t$

$$\Rightarrow 2dx = dt$$

$$I = e^3 \int e^t \left[\frac{t-2}{t^3} \right] \frac{dt}{2}$$

$$= \int e^3 e^t \left[f(t) + f'(t) \right] \frac{dt}{2}$$

$$= \frac{e^3}{2} \cdot e^t \cdot f(t) + C$$

$$= \frac{e^3}{2} e^{2x-3} \cdot \frac{1}{(2x-3)^2} + C$$

$$= \frac{e^{2x}}{2(2x-3)^2} + C$$



OR

Find: $\int \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} dx$.

Answer:

$$I = \int \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} dx$$

$$\text{Let } \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)}$$

$$= \frac{A}{(x + 2)} + \frac{Bx + C}{(x^2 + 1)}$$

$$= \frac{A(x^2 + 1) + (Bx + C)(x + 2)}{(x + 2)(x^2 + 1)}$$

$$= \frac{(A + B)x^2 + (2Bx + C)x + (2C + A)}{(x + 2)(x^2 + 1)}$$

$$\therefore A + B = 1 \dots\dots(i)$$

$$2B + C = 1 \dots\dots(ii)$$

$$2C + A = 1 \dots\dots(iii)$$

$$\Rightarrow A = 1 - 2C \dots\dots(iv)$$

$$\text{From (ii) } B = \frac{1 - C}{2} \dots\dots(v)$$

Put (4) and (5) in (1) we get.

$$1 - 2C + \frac{1 - C}{2} = 1$$

$$\Rightarrow 2 - 4C + 1 - C = 2$$

$$\Rightarrow 5C = 1$$

$$\Rightarrow C = \frac{1}{5}$$

$$\therefore A = 1 - 2C = 1 - \frac{2}{5} = \frac{3}{5}$$

$$B = \frac{1 - C}{2} = \frac{1 - \frac{1}{5}}{2} = \frac{\frac{4}{5}}{2} = \frac{2}{5}$$

$$\therefore \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} = \frac{\frac{3}{5}}{(x + 2)} + \frac{\frac{2}{5}x + \frac{1}{5}}{(x^2 + 1)}$$

$$\therefore I = \int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx$$

$$\Rightarrow I = \int \frac{\frac{3}{5}}{(x + 2)} + \frac{\frac{2}{5}x + \frac{1}{5}}{(x^2 + 1)} dx$$

$$\Rightarrow I = \frac{3}{5} \log(x + 2) + \frac{1}{5} \left[\int \frac{2x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx \right]$$

$$\Rightarrow I = \frac{3}{5} \log(x + 2) + \frac{1}{5} (x^2 + 1) + \tan^{-1} x + C$$



Question: 13

Evaluate: $\int_{-2}^2 \frac{x^2}{1+5^x} dx$

Answer:

$$I = \int_{-2}^2 \frac{x^2}{1+5^x} dx \quad \dots\dots\dots (i)$$

Using Property $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$I = \int_{-2}^2 \frac{(-x)^2}{1+5^{-x}} dx$$

$$\Rightarrow I = \int_{-2}^2 \frac{5^x x^2}{1+5^x} dx \quad \dots\dots\dots (ii)$$

Add (1) and (2), we get

$$2I = \int_{-2}^2 \frac{1+5^x}{1+5^x} x^2 dx$$

$$= \int_{-2}^2 x^2 dx$$

$$= \left[\frac{x^3}{3} \right]_{-2}^2$$

$$= \frac{1}{3} [2^3 - (-2)^3]$$

$$= \frac{1}{3} [8 - (-8)]$$

$$2I = \frac{16}{3} \quad \therefore I = \frac{8}{3}$$

Question: 14

Find: $\int (x+3)\sqrt{3-4x-x^2} dx$

Answer:

$$I = \int (x+3)\sqrt{3-4x-x^2} dx$$

$$\text{Let } x+3 = \lambda \frac{d}{dx} (3-4x-x^2) + \mu$$

$$\Rightarrow x+3 = \lambda(-2x-4) + \mu$$

$$\Rightarrow x+3 = -2\lambda x - 4\lambda + \mu$$

$$\therefore -2\lambda = 1$$

$$\Rightarrow \lambda = -\frac{1}{2}$$

$$-4\lambda + \mu = 3$$



$$\begin{aligned}
&\Rightarrow -4\left(-\frac{1}{2}\right) + \mu = 3 \\
&\Rightarrow 2 + \mu = 3 \\
&\Rightarrow \mu = 1 \\
\therefore I &= \int \left[-\frac{1}{2} \frac{d}{dx} (3 - 4x - x^2) + 1 \right] \sqrt{3 - 4x - x^2} dx \\
&= -\frac{1}{2} \int \frac{d}{dx} (3 - 4x - x^2) \sqrt{3 - 4x - x^2} dx + \int \sqrt{3 - 4x - x^2} dx \\
&= -\frac{1}{2} \frac{(3 - 4x - x^2)^{\frac{3}{2}}}{\frac{3}{2}} + \int \sqrt{3 - x^2 - 4x - 4 + 4} dx \\
&= -\frac{(3 - 4x - x^2)^{\frac{3}{2}}}{3} + \int \sqrt{7 - (x + 2)^2} dx \\
&= -\frac{(3 - 4x - x^2)^{\frac{3}{2}}}{3} + \frac{x + 2}{2} \sqrt{7 - (x + 2)^2} \sin^{-1} \left(\frac{x + 2}{\sqrt{7}} \right) + C
\end{aligned}$$

Question: 15

Find the particular solution of differential equation: $\frac{dy}{dx} = \frac{x + y \cos x}{1 + \sin x}$ given that $y = 1$ when $x = 0$.

Answer:

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{x + y \cos x}{1 + \sin x} \\
\Rightarrow \frac{dy}{dx} + \frac{\cos x}{1 + \sin x} y &= -\frac{x}{1 + \sin x} \dots\dots\dots (i)
\end{aligned}$$

This is a linear differential equation with

$$P = \frac{\cos x}{1 + \sin x} \quad Q = \frac{-x}{1 + \sin x}$$

$$\begin{aligned}
\therefore \text{I.F.} &= e^{\int \frac{\cos x}{1 + \sin x} dx} \\
&= e^{\log(1 + \sin x)} \\
&= (1 + \sin x)
\end{aligned}$$

Multiplying both sides of (i) by I.F. = $1 + \sin x$, we get

$$(1 + \sin x) \frac{dy}{dx} + y \cos x = -x$$

Integrating with respect to x , we get $y(1 + \sin x) = \int -x dx + C$

$$\Rightarrow y = \frac{2C - x^2}{2(1 + \sin x)} \dots\dots\dots (ii)$$

Given that $y = 1$ when $x = 0$

$$\begin{aligned}
\therefore 1 &= \frac{2C}{2(1 + 0)} \\
\Rightarrow C &= 1 \dots\dots\dots (iii)
\end{aligned}$$

$$\therefore \text{Put (3) in (2), we get } y = \frac{2 - x^2}{2(1 + \sin x)}$$



Question: 16

Find the particular solution of the differential equation $2y e^{\frac{x}{y}} dx + \left(y - 2x e^{\frac{x}{y}} \right) dy = 0$ given that $x = 0$ when $y = 1$.

Answer:

$$2y e^{\frac{x}{y}} dx + \left(y - 2x e^{\frac{x}{y}} \right) dy = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{2x e^{\frac{x}{y}} - y}{2y e^{\frac{x}{y}}}$$

The given D.E. is a homogeneous differential equation.

\therefore Put $x = vy$

$$\frac{dx}{dy} = v + y \frac{dv}{dy}$$

$$v + y \frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$\Rightarrow y \frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v$$

$$\Rightarrow y \frac{dv}{dy} = -\frac{1}{2e^v}$$

$$\Rightarrow 2ye^v dv = -dy$$

$$\Rightarrow 2e^v dv = -\frac{1}{y} dy$$

$$\Rightarrow 2 \int e^v dv = - \int \frac{1}{y} dy$$

$$\Rightarrow 2e^v = -\log |y| + \log C$$

$$\Rightarrow 2e^v = \log \left| \frac{C}{y} \right|$$

$$\Rightarrow 2e^{\frac{x}{y}} = \log \left| \frac{C}{y} \right|$$

Given that at $x = 0$, $y = 1$.

$$\therefore 2 \cdot e^0 = \log \left| \frac{C}{1} \right|$$

$$\Rightarrow C = e^2$$

$$\therefore 2e^{\frac{x}{y}} = \log \frac{e^2}{y}$$

Question: 17

Show that the four points A(4, 5, 1), B(0, -1, -1), C(3, 9, 4) and D(-4, 4, 4) are coplanar.

Answer:

A (4, 5, 1) B (0, -1, -1) C (3, 9, 4) D (-4, 4, 4)

$$\overrightarrow{AB} = (-4\hat{i} - 6\hat{j} - 2\hat{k})$$

$$\overrightarrow{AC} = (-\hat{i} - 4\hat{j} + 3\hat{k})$$

$$\overrightarrow{AD} = (-8\hat{i} + \hat{j} + 3\hat{k})$$

$$\therefore [\overrightarrow{AB} \overrightarrow{AC} \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

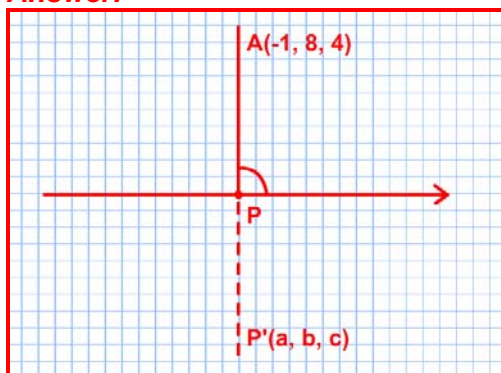
$$\begin{aligned} \therefore -4 &= \begin{vmatrix} 4 & 3 \\ -1 & 3 \end{vmatrix} + 6 \begin{vmatrix} -1 & 3 \\ -8 & 3 \end{vmatrix} - 2 \begin{vmatrix} -1 & 4 \\ -8 & -1 \end{vmatrix} \\ &= -4(12 + 3) + 6(-3 + 24) - 2(1 + 32) \\ &= -60 + 126 - 66 \\ &= 0 \end{aligned}$$

\therefore Four points A, B, C, D are coplanar.

Question: 18

Find the coordinates of the foot of perpendicular drawn from the point A(-1, 8, 4) to the line joining the points B(0, -1, 3) and C (2, -3, -1). Hence find the image of the point A in the line BC.

Answer:



$$\text{Equation of Line BC} \Rightarrow \frac{x - x^1}{x^2 - x^1} = \frac{y - y^1}{y^2 - y^1} = \frac{z - z^1}{z^2 - z^1}$$

$$\frac{x - 0}{2} = \frac{y + 1}{-2} = \frac{z - 3}{-4} = \lambda$$

general coordinates of P

$$P(2\lambda, -2\lambda - 1, -4\lambda + 3)$$

$$\text{D.R of AP } (2\lambda + 1, -2\lambda - 9, -4\lambda - 1)$$

$$AP \perp BC$$

$$2(2\lambda + 1) - 2(-2\lambda - 9) - 4(-4\lambda - 1) = 0$$

$$4\lambda + 2 + 4\lambda + 18 + 16\lambda + 4 = 0$$

$$24 + 24\lambda = 0$$

$$\lambda = -1$$

$$P(-2, 1, 7)$$

Coordinates of foot of \perp (-2, 1, 7)

Coordinates of image of A is P' (a, b, c) is

$$\frac{a - 1}{2} = -2, a = -3$$



$$\frac{b+8}{2} = 1, b = -6$$

$$\frac{c+4}{2} = 7, c = 10$$

$$P'(-3, -6, 10)$$

Question: 19

A bag X contains 4 white balls and 2 black balls, while another bag Y contains 3 white balls and 3 black balls. Two balls are drawn (without replacement) at random from one of the bags and were found to be one white and one black. Find the probability that the balls were drawn from bag Y.

Answer:

bag A = 4 white, 2 black

bag y = 3 white, 3 black

E_1 = first bag selected

E_2 = second bag selected

$$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}$$

$$P\left(\frac{A}{E_1}\right) = \frac{4}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{4}{5} = \frac{16}{30}$$

$$P\left(\frac{A}{E_2}\right) = \frac{3}{6} \times \frac{3}{5} + \frac{3}{6} \times \frac{3}{5} = \frac{18}{30}$$

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)P\left(\frac{E_2}{A}\right)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$P\left(\frac{E_2}{A}\right) = \frac{\frac{1}{2} \times \frac{18}{30}}{\frac{1}{2} \times \frac{16}{30} + \frac{1}{2} \times \frac{18}{30}}$$

$$= \frac{18}{16+18} = \frac{18}{34} = \frac{9}{17}$$

OR

A and B throw a pair of dice alternately, till one of the m gets a total of 10 and wins the game. Find their respective probabilities of winning, if A starts first.

Answer:

$$P(\text{win}) = \frac{3}{36} = \frac{1}{12}$$

$$P(\text{lose}) = \frac{11}{12}$$

$$P(\text{A wins}) = \frac{1}{12} + \frac{11}{12} \times \frac{11}{12} \times \frac{1}{12} + \left(\frac{11}{12}\right)^4 \times \frac{1}{12} + \dots$$

$$a = \frac{1}{12} \quad r = \frac{121}{144}$$

by using formula of infinite G.P.



$$P(\text{Awins}) = \frac{\frac{1}{12}}{1 - \frac{121}{144}} = \frac{12}{23}$$



Section C (*Question numbers 20 to 26 carry 6 mark each*)



Question: 20

Three numbers are selected at random (without replacement) from first six positive integers. Let X denote the largest of the three numbers obtained. Find the probability distribution of X. Also, find the mean and variance of the distribution.

Answer:

X = larger of three numbers

X = 3, 4, 5, 6

$$P(x = 3) = 6 \times \frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$$

$$P(x = 4) = 18 \times \frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{3}{20}$$

$$P(x = 5) = 36 \times \frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{6}{20}$$

$$P(x = 6) = 60 \times \frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{10}{20}$$

X_i	P_i	$P_i X_i$	$P_i X_i^2$
3	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{9}{20}$
4	$\frac{3}{20}$	$\frac{12}{20}$	$\frac{48}{20}$
5	$\frac{6}{20}$	$\frac{30}{20}$	$\frac{150}{20}$
6	$\frac{10}{20}$	$\frac{60}{20}$	$\frac{360}{20}$

$$\text{Mean} = \sum P_i X_i^2 = \left(\frac{105}{20} \right)^2 = 5.25$$

$$\sum P_i X_i^2 = \frac{567}{20}$$

$$\begin{aligned} \text{Var}(X) &= \sum P_i X_i^2 - \left(\sum P_i X_i \right)^2 \\ &= \frac{567}{20} - \left(\frac{105}{20} \right)^2 = 0.787 \end{aligned}$$

Question: 21

Let $A = R \times R$ and $*$ be a binary operation on A defined by

$$(a, b) * (c, d) = (a+c, b+d)$$

Show that $*$ is commutative and associative. Find the identity element for $*$ on A. Also find the inverse of every element $(a, b) \in A$.

Answer:

$$(a, b) * (c, d) = (a + c, b + d)$$

i. Commutative

$$(a, b) * (c, d) = (a+c, b+d)$$

$$(c, d) * (a, b) = (c+a, d+b)$$



for all, $a, b, c, d \in R$
 $*$ is commulative on A

ii. Associative : _____
 $(a, b), (e, d), (e, f) \in A$
 $\{ (a, b) * (c, d) \} * (e, f)$
 $= (a + c, b + d) * (e, f)$
 $= ((a + c) + e, (b + d) + f)$
 $= (a + (c + e), b + (d + f))$
 $= (a * b) * (c + d, d + f)$
 $= (a * b) \{ (c, a) * (e, f) \}$
 is associative on A

Let (x, y) be the identity element in A.
 then,

$(a, b) * (x, y) = (a, b)$ for all $(a, b) \in A$
 $(a + x, b + y) = (a, b)$ for all $(a, b) \in A$
 $a + x = a, b + y = b$ for all $(a, b) \in A$
 $x = 0, y = 0$
 $(0, 0) \in A$
 $(0, 0)$ is the identity element in A.

Let (a, b) be an invertible element of A.

$(a, b) * (c, d) = (0, 0) = (c, d) * (a, b)$
 $(a + c, b + d) = (0, 0) = (c + a, d + b)$
 $a + c = 0 \quad b + d = 0$
 $a = -c \quad b = -d$
 $c = -a \quad d = -b$
 (a, b) is an invertible element of A, in such a case the inverse of (a, b) is $(-a, -b)$

Question: 22

Prove that $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$ is an increasing function of θ on

Answer:

$$y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$$

$$\frac{dy}{d\theta} = \frac{(2 + \cos \theta)(4 \cos \theta) + 4 \sin^2 \theta}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4}{(2 + \cos \theta)^2} - 1$$

$$= \frac{4 \cos \theta - \cos^2 \theta}{(2 + \cos \theta)^2}$$

$$\frac{dy}{d\theta} = \frac{\cos \theta (4 - \cos \theta)}{(2 - \cos \theta)^2}$$



for increasing $\frac{dy}{d\theta} > 0$

$$\theta \in \left(0, \frac{\pi}{2}\right)$$

$$0 \leq \cos \leq 1$$

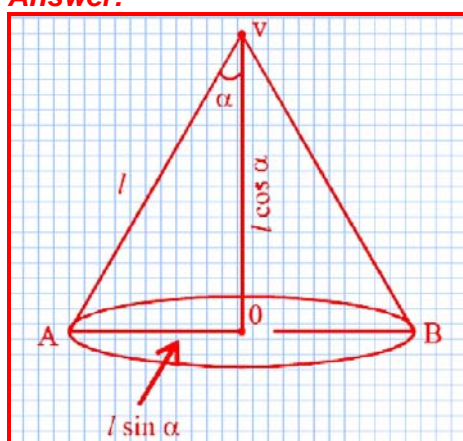
$(2 + \cos\theta)^2$ always greater than 0

So, $\frac{dy}{d\theta}$ is increasing on $\left[0, \frac{\pi}{2}\right]$

OR

Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left[0, \frac{\pi}{2}\right]$

Answer:



$$\begin{aligned}\text{Volume of cone} &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi (\ell \sin \alpha)^2 (\ell \cos \alpha) \\ &= \frac{1}{3} \pi \ell^3 \sin^2 \alpha \cos \alpha\end{aligned}$$

$$\begin{aligned}\frac{dv}{d\alpha} &= \frac{\pi \ell^3}{3} [-\sin^3 \alpha + 2 \sin \alpha \cos \alpha \times \cos \alpha] \\ &= \frac{\pi \ell^3 \sin \alpha}{3} (-\sin^2 \alpha + 2 \cos^2 \alpha)\end{aligned}$$

For maximum or minimum $\frac{dv}{d\alpha} = 0$

$$\frac{\pi \ell^3 \sin \alpha}{3} (-\sin^2 \alpha + 2 \cos^2 \alpha) = 0$$

$$\sin \alpha \neq 0$$

$$2 \cos^2 \alpha = \sin^2 \alpha$$

$$\tan^2 \alpha = 2$$



$$\tan \alpha = \sqrt{2}$$

$$\cos \alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \cos^{-1} \frac{1}{\sqrt{3}}$$

Again diff. w.r.t. α , we get

$$\frac{d^2v}{d\alpha^2} = \frac{1}{3} \pi \ell^3 \cos^2 \alpha (2 - 7 \tan^2 \alpha)$$

$$\text{At } \cos \alpha = \frac{1}{\sqrt{3}}$$

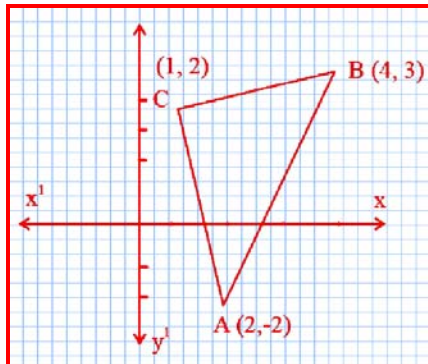
$$\frac{d^2v}{d\alpha^2} < 0$$

V is maximum when $\cos \alpha = \frac{1}{\sqrt{3}}$ or $\alpha = \cos^{-1} \frac{1}{\sqrt{3}}$

Question: 23

Using the method of integration, find the area of the triangular region whose vertices are $(2, -2)$, $(4, 3)$ and $(1, 2)$.

Answer:



$$\text{Equation of line AB } y + 2 = \frac{3 + 2}{2}(x - 2)$$

$$\Rightarrow 2y = 5x - 14$$

$$\text{Equation of line BC } y - 3 = \frac{1}{3}(x - 4)$$

$$\Rightarrow 3y = x + 5$$

$$\text{Equation of line CA } (y - 2) = -4(x - 1)$$

$$4x + y = 6$$

$\therefore \text{ar}(\triangle ABC)$

$$= \int_{-2}^3 \frac{2y + 14}{5} dy - \int_2^3 3y - 5 dy - \int_{-2}^2 \frac{6 - y}{4} dy$$

$$= \frac{75}{5} - \frac{5}{2} - \frac{24}{4}$$



$$= \frac{300 - 120 - 50}{20} = \frac{130}{20}$$

$$= \frac{13}{2} \text{ sq. units}$$

Question: 24

Find the equation of the plane which contains the line of intersection of the planes

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0 \text{ and}$$

$$\vec{r} \cdot (-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$$

and whose intercept on x-axis is equal to that of on y-axis.

Answer:

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0$$

$$\vec{r} \cdot (-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$$

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda \{ \vec{r} \cdot (-2\hat{i} + \hat{j} + \hat{k}) \} - 4 + 5\lambda = 0$$

$$\Rightarrow \vec{r} \cdot [(1 - 2\lambda)\hat{i} + (-2 + \lambda)\hat{j} + (3 + \lambda)\hat{k}] - 4 + 5\lambda = 0$$

$$\Rightarrow (1 - 2\lambda)x + (-2 + \lambda)y + (3 + \lambda)z = -5\lambda + 4$$

$$\Rightarrow \frac{x}{\frac{-5\lambda + 4}{1 - 2\lambda}} + \frac{y}{\frac{-5\lambda + 4}{-2 + \lambda}} + \frac{z}{\frac{-5\lambda + 4}{3 + \lambda}} = 1$$

$$\Rightarrow \frac{-5\lambda + 4}{1 - 2\lambda} = \frac{-5\lambda + 4}{-2 + \lambda}$$

$$\Rightarrow 1 - 2\lambda = -2 + \lambda$$

$$\Rightarrow -3\lambda = -3$$

$$\Rightarrow \lambda = 1$$

\therefore Equation of the required plane

$$-x - y + 4z = -1$$

$$x + y - 4z - 1 = 0$$

Vector equation of the required Plane

$$\vec{r} \cdot (\hat{i} + \hat{j} - 4\hat{k}) - 1 = 0$$

Question: 25

A retired person wants to invest an amount of Rs. 50, 000. His broker recommends investing in two type of bonds 'A' and 'B' yielding 10% and 9% return respectively on the invested amount. He decides to invest at least Rs.20,000 in bond 'A' and at least Rs. 10,000 in bond 'B'. He also wants to invest at least as much in bond 'A' as in bond 'B'. Solve this linear programming problem graphically to maximise his returns.

Answer:

$$Z = 0.1x + 0.09y$$

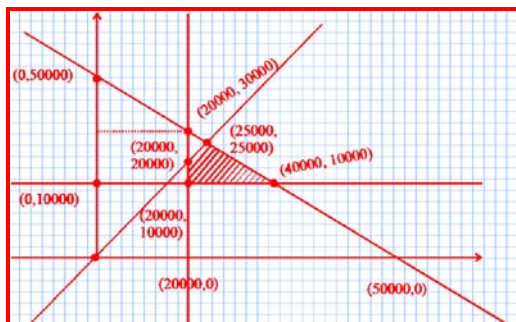
$$x + y \leq 50000$$

$$x \geq 20000$$

$$y \geq 10000$$

$$y \leq x$$





	$z = 0.1x + 0.09y$
$P_1(20000, 10000)$	2900
$P_2(40000, 10000)$	4900
$P_3(25000, 25000)$	4750
$P_4(20000, 20000)$	3800

When A invest 40000 & B invest 10000

Question: 26

Using properties of determinants, prove that

$$\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix} = 2xyz(x+y+z)^3$$

Answer:

$$\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix} = 2xyz(x+y+z)^3$$

L.H.S.

Multiplying R_1 , R_2 and R_3 by z , x , y respectively

$$= \frac{1}{xyz} \begin{vmatrix} z(x+y)^2 & z^2x & z^2y \\ x^2z & x(z+y)^2 & x^2y \\ y^2z & xy^2 & y(z+x)^2 \end{vmatrix}$$

Take common z , x , y from C_1 , C_2 , & C_3

$$= \frac{xyz}{xyz} \begin{vmatrix} (x+y)^2 & z^2 & z^2 \\ x^2 & (z+y)^2 & x^2 \\ y^2 & y^2 & (z+x)^2 \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_3 \text{ and } C_2 \rightarrow C_2 - C_3$$

$$= \frac{c+4}{2} = 7$$

Taking common $x+y+z$ from C_1 & C_2



$$= (x+y+z)^2 \begin{vmatrix} x+y-z & 0 & z^2 \\ 0 & z+y-x & x^2 \\ y-z-x & y-z-x & (z+x)^2 \end{vmatrix}$$

$$R_3 \rightarrow R_3 - (R_1 + R_2)$$

$$= (x+y+z)^2 \begin{vmatrix} x+y-z & 0 & z^2 \\ 0 & z+y-x & x^2 \\ -2x & -2z & 2xz \end{vmatrix}$$

$$C_1 \rightarrow zC_1$$

$$C_2 \rightarrow xC_2$$

$$= \frac{(x+y+z)^2}{xz} \begin{vmatrix} z(x+y) & 0 & z^2 \\ 0 & x(z+y-x) & x^2 \\ -2xz & -2xz & 2xz \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_3$$

$$C_2 \rightarrow C_2 + C_3$$

$$= \frac{(x+y+z)^2}{xz} \begin{vmatrix} z(x+y) & 0 & z^2 \\ x^2 & x(z+y) & x^2 \\ 0 & 0 & 2xz \end{vmatrix}$$

Taking z and x common from R_1 & R_2

$$= \frac{(x+y+z)^2}{xz} x \cdot z x \begin{vmatrix} x+y & z & z \\ x & z+y & x \\ 0 & 0 & 2xz \end{vmatrix}$$

expansion along R_3

$$\begin{aligned} &= (x+y+z)^2 \times 2xz ((x+y)(z+y) - xz) \\ &= (x+y+z)^2 \times 2xz (xz + xy + yz + y^2 - xz) \\ &= (x+y+z)^2 \times 2xz (xy + yz + y^2) \\ &= 2xyz (x+y+z)^3 \end{aligned}$$

OR

$$\text{If } A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \text{ and } A^3 - 6A^2 + 7A + kI^3 = O \text{ find } k.$$

Answer:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$



$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 7A + KI_3 = 0$$

$$\Rightarrow \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow k = 2$$

